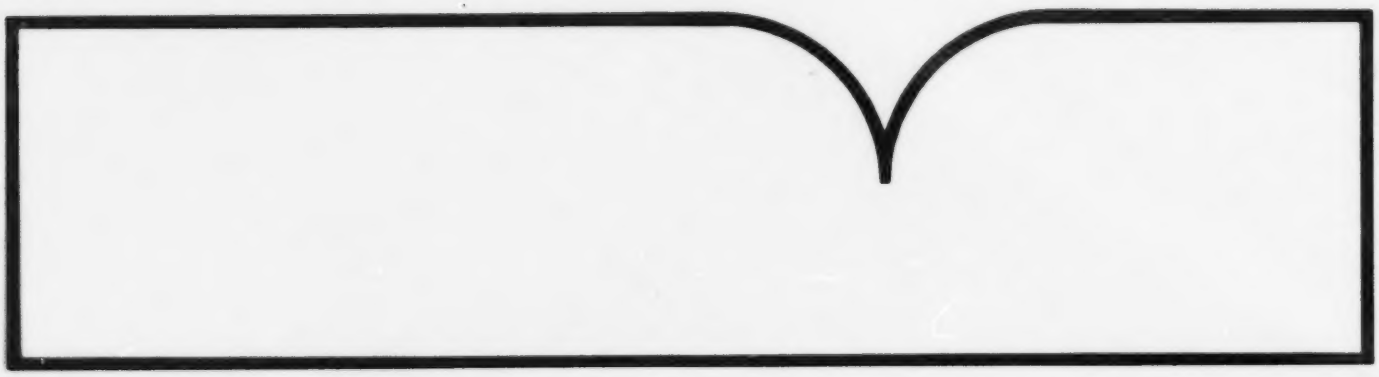
U.S. DEPARTMENT OF COMMERCE

PB80-163058

Turbulent Diffusion from a Point Source in Stratified and Neutral Flows Around a Three-Dimensional Hill-Part I Diffusion Equation Analysis

(U.S.) Environmental Sciences Research Lab.
Research Triangle Park, NC

1979



U.S. Department of Commerce
National Technical Information Service
NTIS.

BIBLIOGRAPHIC INFORMATION

PB80-163058

Turbulent Diffusion from a Point Source in Stratified and Neutral Flows Around a Three-Dimensional Hill-Part I. Diffusion Equation Analysis.

1979

W. H. Snyder, J. C. R. Hunt, and J. S. Puttock.

PERFORMER: Environmental Sciences Research Lab., Research Triangle Park, NC. Meteorology and Assessment Div.
EPA-600/J-79-048

Prepared in cooperation with Atmospheric Environment Service, Downsview (Ontario), and Cambridge Univ. (England). Dept. of Applied Mathematics and Theoretical Physics. Pub. in Journal of Atmospheric Environment, v13, p1227-1239 1979.
In stable stratified flow around three dimensional hills, vertical motion and vertical diffusion is negligible. Consequently turbulent diffusion from a point source can be modelled by considering horizontal flow and horizontal diffusion. Using an eddy diffusivity, the advective diffusive equation around a three dimensional hill which is axisymmetric about a vertical axis is solved to show how source positions on and off the center line affect the trajectories and splitting of impinging plumes and the value and position of the maximum surface concentration on the hill. In the second part of the paper a plume is analyzed in a neutrally stable potential flow around an axisymmetric obstacle such as a hemisphere, also using the diffusion equation. The solutions show how, because streamlines approach the surface of a 3-dimensional hill much more closely than the surface of a 2-dimensional hill, the maximum surface concentrations on the hill can become very much greater than in the absence of the hill. But this only occurs for a limited range of source heights.

KEYWORDS: *Air pollution, *Turbulent flow, *Mathematical models, *Foreign technology, *Atmospheric dispersion.

Available from the National Technical Information Service, Springfield, Va. 22161

PRICE CODE: PC A02/MF A01

Atmospheric Environment Vol. 13, pp. 1227–1239
Pergamon Press Ltd. 1979. Printed in Great Britain

PB80-163705

0004-6981/79/0901-1227 $02.00/0
EPA-600/J-79-048
JOURNAL ARTICLE

TURBULENT DIFFUSION FROM A POINT SOURCE IN STRATIFIED AND NEUTRAL FLOWS AROUND A THREE-DIMENSIONAL HILL—PART I. DIFFUSION EQUATION ANALYSIS

J. C. R. HUNT
Department of Applied Mathematics and Theoretical Physics, Cambridge, CB3 9EW, U.K.

J. S. PUTTOCK*
Atmospheric Environment Service, Downsview, Ontario, Canada, M3H 5T4

and

W. H. SNYDER†
Meteorological and Assessment Division, Environmental Sciences Research Laboratory, U.S. Environmental Protection Agency, Research Triangle Park, NC 27711, U.S.A.

(*First received* 12 *October* 1978 *and in final form* 21 *March* 1979)

Abstract – In stable stratified flow around three dimensional hills vertical motion and vertical diffusion is negligible. Consequently turbulent diffusion from a point source can be modelled by considering horizontal flow and horizontal diffusion. Using an eddy diffusivity the advective diffusive equation around a three dimensional hill which is axisymmetric about a vertical axis is solved to show how source positions on and off the centre line affect the trajectories and splitting of impinging plumes and the value and position of the maximum surface concentration on the hill.

In the second part of the paper we analyse a plume in a neutrally stable potential flow around an axisymmetric obstacle such as a hemisphere, also using the diffusion equation. The solutions show how, because streamlines approach the surface of a 3-dimensional hill much more closely than the surface of a 2-dimensional hill, the maximum surface concentrations on the hill *can* become very much greater than in the absence of the hill. But this only occurs for a limited range of source heights.

1. INTRODUCTION

The prediction of concentration distribution from sources of air pollution in or near hilly terrain is a problem of great practical interest. One method of establishing realistic estimates of diffusion in these very complicated flows is to develop simple mathematical analyses that are based on straightforward physical hypotheses of the flow patterns and turbulent diffusion around simple shapes of hills and then to test these predictions in the laboratory, and against observations in the atmosphere. The alternative approach is to solve hugely complex systems of semi-empirical equations that describe turbulent flows and turbulent diffusion, which are subject to the boundary conditions of complex terrain. Such an approach seems premature at this stage of our understanding.

The particular emphasis of the two parts of this paper is on diffusion around three-dimensional hills when the upwind flow is neutrally or stably stratified. In Pt. I we develop analyses for diffusion from sources near hills both in very stable and neutral conditions. In Pt. II we describe laboratory measurements of the concentration distributions on the surface when a plume impinges onto the hill from an upwind source, and when the stability of the upwind flow varies between very stable and neutral.

In a previous paper we have described the structure of these flows as the stability varies from highly stable to neutral by examining the physical principles governing these flows and by making extensive observations and measurements in a stratified water towing tank and in a wind tunnel, [see Hunt, Snyder and Lawson (1978) or Hunt and Snyder (1979) (referred to hereafter as HS)]. Previous work on very stable flows has been performed by Brighton (1978) and Riley *et al.* (1976). Figures 1(a) and 1(b) indicate some of the differences between very stable and neutral conditions.

The specific aims of the calculations and experiments reported here are:

(a) To investigate the two rival hypotheses about the maximum surface concentration, C_{ms}, in stably stratified flow; these are (i) the EPA valley model (Burt and Slater, 1977), where, effectively, C_{ms} is assumed to be *twice* the centreline concentration, $C^{(\circ)}$, of the plume in the absence of the hill, and (ii) the model of Hunt and Mulhearn (1973), where C_{ms} is approx *equal* to $C^{(\circ)}$. [Note that the *average* concentration actually pre-

* Now at Shell Research Ltd, Thornton Research Centre, P.O. Box 1, Chester CH1 3SH, U.K.

† On assignment from the National Oceanic and Atmospheric Administration, U.S. Department of Commerce.

REPRODUCED BY
NATIONAL TECHNICAL
INFORMATION SERVICE
U. S. DEPARTMENT OF COMMERCE
SPRINGFIELD, VA. 22161

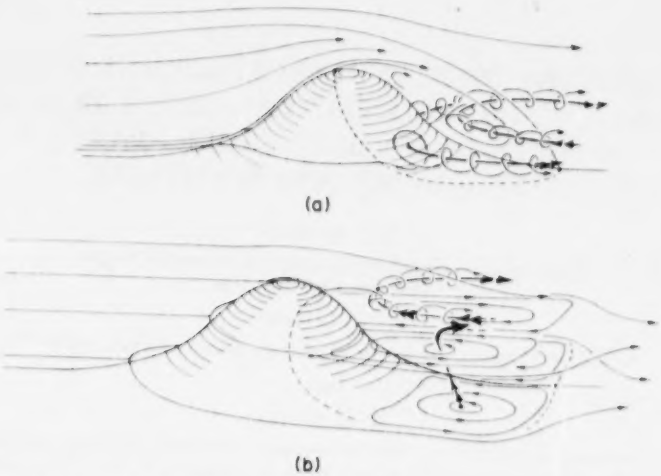


Fig. 1. Sketches of flow over 3-dimensional hills in (a) neutral and (b) highly stratified flows. →→ denotes the vortex lines in the centre of a strongly rotational flow.

dicted (for instance over a 24 h period) may depend more on the assumptions about the crosswind statistics (i.e. the plume direction) than on the precise value of the ratio $C_{mx}/C^{(\infty)}$.] Note also that for neutral turbulent flow round a cylinder or a sphere, even for very thin plumes, wind tunnel observations and calculations show that $C_{mx} \simeq C^{(\infty)}$ for point or line sources on the centreline more than 2 diameters upstream. (Britter, Hunt and Puttock, 1976).

(b) To measure and calculate the effects of small lateral displacements of the source from the centreline of the hill. (This is particularly important for understanding the effects of low frequency lateral wind velocity fluctuations on the average surface concentrations.)

(c) To develop mathematical models for and to measure the maximum surface concentrations on a three-dimension hill in neutral flow and to compare three-dimensional hill in neutral and stably stratified flow and to compare these with those on level ground and on two-dimensional hills. Wind tunnel concentration measurements on two-dimensional hills for upstream sources by Britter and Hunt (to be published), have shown that the effect of the hill is small, whereas, in a model study of complex terrain by Barratt (1973), the three dimensionality seemed to produce a factor of two increase. This difference needs clarification.

In this paper, we do not discuss dispersion from sources on/or downstream of three-dimensional hills. For two-dimensional hills, Huber *et al.* (1976) and Britter, Hunt and Puttock (1976) have shown that the downwash can increase ground level concentrations considerably, through effective reduction of the source height by an amount that is a substantial fraction of the height of the hill. For some recent full scale studies of this latter effect, see Eliseev (1975).

Numerous features of the flow over a hill affect the resulting ground level concentrations. The displacement of the mean streamlines determines how near to the surface the centreline of the plume will reach. The convergence and divergence of the streamlines in the directions normal to and, in the case of three-dimensional flows, parallel to the surface affect the plume width: (a) directly, (b) indirectly through their distorting effects on the velocity and density gradients which, in turn, affect the turbulence (see HS, section 4). Other factors affecting the turbulence over the hill are the heating or cooling of the surface and the changes in the surface shear stress and separation. All these effects can either increase or decrease surface concentrations, though some generalizations about diffusion in complex terrain do not seem to admit this possibility (e.g. Start, Ricks and Dickson, 1976. See especially their Fig. 6).

Essential differences exist between two-dimensional and three-dimensional flows that are worth summarizing:

(a) In three-dimensional flows, streamlines from upstream sources can impinge on the hill surface along a three-dimensional attachment line, and other streamlines can leave along a separation line. In two-dimensional flows, only the separation streamline leaves and impinges on the surface.

(b) Those streamlines in the three-dimensional flow that do not impinge on the surface approach the surface more closely than in two-dimensional flow [see HS, section 3.4, or Egan (1975)].

(c) In 3-dimensional separated flows, external mean streamlines can actually enter the separated swirling flow region, thus advecting pollutants directly into it (Fig. 1b). (Hunt, Abell, Peterka and Woo, 1978), whereas 2-dimensional separated flows have regions of closed mean streamlines into which pollutants reach only by turbulent diffusion.

2. IMPINGING PLUMES AT LOW FROUDE NUMBER

2.1 *General features of the flow*

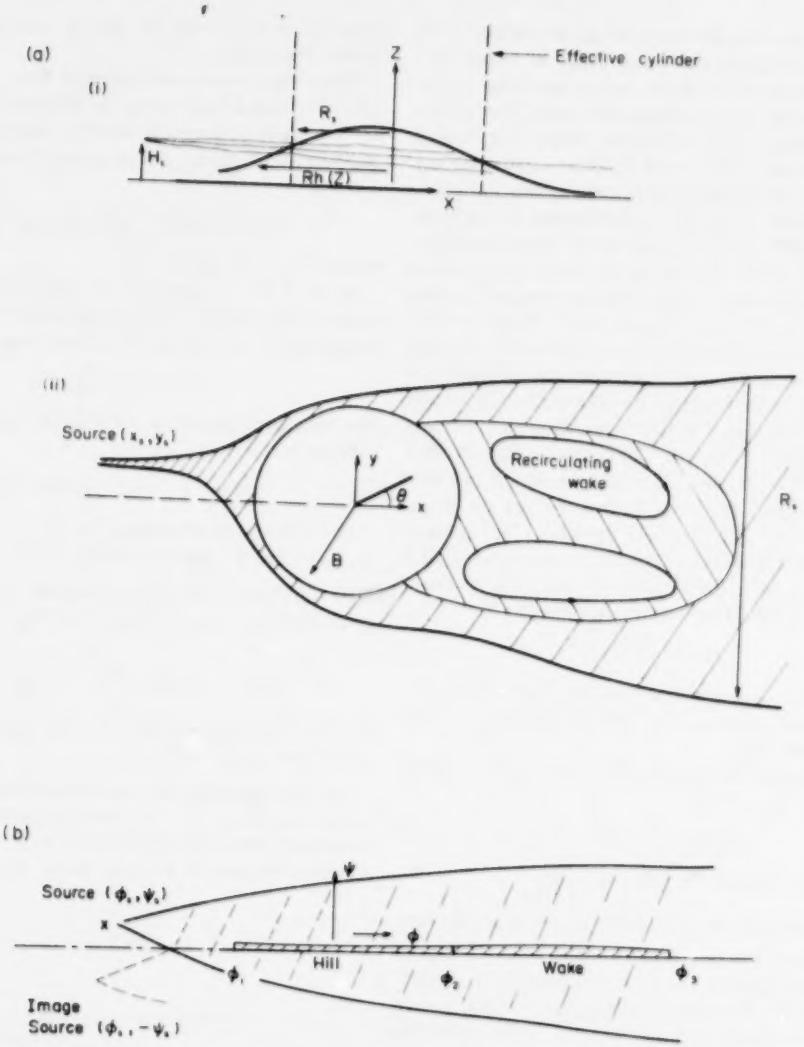


Fig. 2. Stages in the simplification of the analysis of upstream plumes impinging onto 3-dimensional hills when the Froude number F is very small. (a) Side view of impinging plume, showing the equivalent cylinder problem. (b) Section through the hill, the plume and the wake at $z = H_s$. (c) Transformation of the flow around the cylinder in the physical x–y plane into the velocity potential – stream function $\phi - \psi$ planes. ϕ_1, ϕ_2, ϕ_3 correspond to the values of ϕ at the stagnation point, separation point and the rear of the wake recirculation region.

Consider a uniform steady stream or air flow with velocity U_s in a uniform stable density gradient $(-\partial\rho/\partial z)$, flowing round a hill with height h, then the flow structure at high enough Reynolds number is determined by the Froude number

$$F = \frac{U_s}{Nh}, \quad \text{where } N = \sqrt{(g/\rho)(-\partial\rho/\partial z)}.$$

The Froude number can be thought of most usefully as the ratio of inertial to gravitational forces in the flow.

If F is sufficiently small, theoretical arguments (Drazin, 1961) and the experiments of HS, Brighton (1978) and Riley *et al.* (1976) indicate that the flow around any three-dimensional hill is horizontal for $z < [H(1 - F)]$. (See Figs. 1, 2a.) What happens to a turbulent plume introduced into this laminar flow at a height $H_s < h(1 - F)$? In HS, the mean streamline through the source was shown to impinge on the hill and therefore the plume must do so as well.* The

* Many field observations confirm this result, for example those by Dr. Venter of chimney plumes in Natal (see Brighton, 1978) or those reported by Burt and Slater (1977).

concentration around the point on the surface where the plume impinges can, in principle, be computed if the turbulence in the plume can be described statistically, using the theory developed by Hunt (1973), Hunt and Mulhearn (1973) and Hunt, Britter and Puttock (1979). However, Hunt and Mulhearn's analysis suggests that the plume development (given the plume width upwind of the hill), is determined primarily by the divergence and convergence of the streamlines.† This effect of the streamlines is particularly marked when the turbulence in the plume has decayed to a very low level (say less than 3%) as it assuredly does after a few kilometers of travel, or in the experiment described in Pt. II in about 1 m. Consequently, as Hunt and Mulhearn demonstrated quantitatively, the plume width and concentration can be estimated by assuming (a) a constant diffusivity, D^*, in the *horizontal plane* where $D^*/(U_{H_s}R_s) \ll 1$, R_s being the typical radius of the hill at the source height and (b) that the plume thickness in the vertical direction σ_z remains effectively constant as the plume passes round the hill, so that vertical diffusion can be ignored. The necessary conditions for this simplification are that

(i) $$(\sigma_w/U_{H_s})R_s \ll \sigma_z^{(1)}, \tag{1}$$

if $\sigma_z^{(1)}$ is the vertical plume thickness in the absence of the hill, and σ_w is the r.m.s. vertical turbulence velocity component, and

(ii) the vertical divergence of the streamlines is small i.e. (from 2.2 of HS)

$$F^2(\mathrm{d}^2R_s/\mathrm{d}z^2)\sigma_z^{(1)} \ll 1, \tag{2}$$

where the slope of the hill is $(\mathrm{d}R_s/\mathrm{d}z)^{-1}$.

2.2 *Model solution: line source upwind of a cylinder*

In a flow with low Froude number the problem of plume impingement has now been reduced to calculating the two dimensional diffusion of a *line* source around a *cylinder* with axis parallel to the z-axis, the cylinder having the same cross section as the hill at the height H_s. For an axisymmetric hill with radius $R_s(z)$, the cylinder is circular with radius $R_s = R_s(z = H_s)$ see Figs. 2(a), 2(b). The horizontal flow around the cylinder is potential flow outside the wake since the turbulence is negligible and since the surface does not influence the velocity above it. Normalising the concentration C on the source strength Q, the hill radius R_s, $\sigma_z^{(1)}$ and the upstream velocity at $z = H_s$, (U_{H_s}) and denoting it by χ, we obtain

$$\chi(x, y) = CU_{H_s}R_s\sigma_z^{(1)}/Q. \tag{3}$$

† These effects were recently demonstrated in measurements of a thermal mixing layer in a wind tunnel contraction by Keffer and Kawall. The theory of Hunt and Mulhearn described the results quite well, (Keffer, Kawall, Hunt and Maxey, 1978).

‡ This would correspond in a very stable atmosphere ("F" category) to a plume from a source about 2 km away from a hill with radius 1 km, where $\sigma_z \simeq 80$ m.

(The variation of χ with z on the hill must be the same as that upstream.)

Given two-dimensional potential flow, a constant diffusivity, and a line source, the diffusion equation is best written in terms of the velocity stream function ψ and velocity potential ϕ (normalised in terms of U_{H_s} and R_s),

$$\partial\chi/\partial\phi = D[\partial^2\chi/\partial\phi^2 + \partial^2\chi/\partial\psi^2], \tag{4}$$

where $D = D^*/[U_{H_s}R_s]$.

Since $D \ll 1$, (typically for the plumes in our experiment $D = 0.001$)‡ the plume width is very small compared to the downwind distance, and therefore

$$|\partial/\partial\psi| \gg |\partial/\partial\phi|. \tag{5}$$

The boundary condition on χ at the surface of the cylinder is

$$\partial\chi/\partial n = 0 \quad \text{or} \quad \partial\chi/\partial\psi = 0. \tag{6}$$

The solution to (4) subject to (5) for a source at (x_s, y_s) or (ϕ_s, ψ_s) has three components:

(i) *Upstream of the stagnation point* the solution is given by Hunt and Mulhearn (1973) as

$$\chi = \frac{1}{2\{\pi(\phi - \phi_s)D\}^{1/2}}\exp\left[-\frac{(\psi - \psi_s)^2}{4D(\phi - \phi_s)}\right]. \tag{7}$$

This formula also satisfies (6) if the source is on the centre line (i.e. $\psi_s = 0$).

(ii) *Downstream of the stagnation point, but upstream of the wake* $\phi_1 < \phi < \phi_2$ (see Fig. 3), the solution is obtained from the distribution of χ on $\phi = \phi_1$, (7), and the introduction of suitable image distributions to

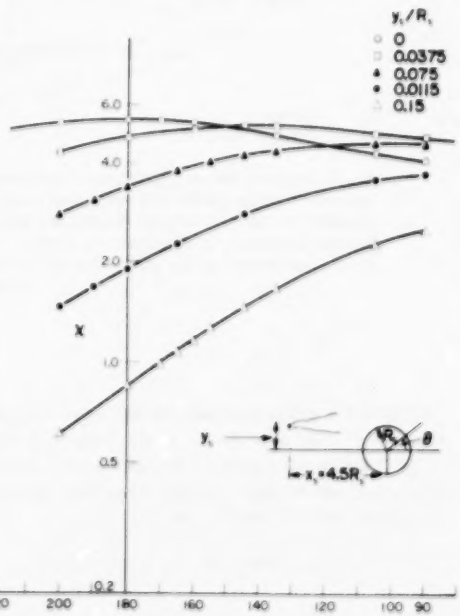


Fig. 3. Calculation of surface concentration around an axisymmetric hill at $z = H_s$, as a function of the source displacement y_s/R_s. Source located at $x = 4.5R_s$.

satisfy the boundary condition (6). (This trick was developed by Puttock and Hunt, 1979.)

Then

$$\chi(\phi,\psi) = \frac{1}{2[\pi(\phi-\phi_1)D]^{1/2}} \int_{L_1}^{L_2} \chi_1(\psi') \times \left[\exp\left(-\frac{(\psi-\psi')^2}{4(\phi-\phi_1)D} \right) + \exp\left(-\frac{(\psi+\psi')^2}{4(\phi-\phi_1)D} \right) \right] d\psi', \tag{8}$$

where for

$$\psi > 0, \quad L_2 = \infty, \quad L_1 = 0$$
$$\psi < 0, \quad L_2 = 0, \quad L_1 = -\infty$$

and $\chi_1(\psi')$ is given by (7) with $\phi = \phi_1$. By inspection (8) is continuous with (7) on $\phi = \phi_1$, but since $\partial\chi/\partial\phi$ is not continuous, it violates the condition (5). However, as Puttock and Hunt, (1979) showed in a similar context, the error introduced by using the parabolic form of (4) is small. A brief argument deriving the error term is given in the appendix. Equation (8) can be restated in terms of known functions as:

$$\chi = \frac{I_1}{4\pi D[(\phi-\phi_1)(\phi_1-\phi_s)]^{1/2}}, \tag{9a}$$

where

$$I_1 = \frac{\sqrt{\pi}}{2} I_2 \exp[-(\psi_s^2/B + \psi^2/A)]/\sqrt{1/A + 1/B}, \tag{9b}$$

$$I_2 = \exp[(\psi_s/B + \psi/A)^2/(1/A + 1/B)] \times (1 \pm \mathrm{erf}[(\psi_s/B + \psi/A)/\sqrt{1/A + 1/B}]) + \exp[(\psi_s/B - \psi/A)^2/(1/A + 1/B)] \times (1 \pm \mathrm{erf}[(\psi_s/B - \psi/A)/\sqrt{1/A + 1/B}]), \tag{9c}$$

concentrations in the wake of a circular cylinder in a turbulent neutrally stable flow with line sources upstream. If $\chi_{2+}(\psi)$, $\chi_{2-}(\chi)$ are the concentration distributions on the line $\phi = \phi_2$ for $\psi > 0$ and $\psi < 0$ respectively (i.e. just outside the wake on each side of the cylinder), then the solution to (4) such that $\chi = \chi_2(\psi)$ on $\phi = \phi_2$, and $\chi = \chi_w$ on $\psi = 0$, is, for $\psi > 0$

$$\chi(\phi,\psi) = \frac{1}{2[\pi(\phi-\phi_2)D]^{1/2}} \times \left[\int_0^\infty \left\{ \exp\left(-\frac{(\psi-\psi')^2}{4D(\phi-\phi_2)} \right) \times \chi_{2+}(\psi') + \exp\left(-\frac{(\psi+\psi')^2}{4D(\phi-\phi_2)} \right) \times (-\chi_{2+}(\psi') + 2\chi_w) \right\} d\psi' \right] \tag{10a}$$

and for $\psi < 0$,

$$\chi(\phi,\psi) = \frac{1}{2[\pi(\phi-\phi_2)D]^{1/2}} \times \left[\int_0^\infty \left\{ \cdot \exp\left(-\frac{(\psi-\psi')^2}{4D(\phi-\phi_2)} \right) \times (-\chi_{2-}(\psi') + 2\chi_w) + \exp\left(-\frac{(\psi+\psi')^2}{4D(\phi-\phi_2)} \right) \times \chi_{2-}(\psi') \right\} d\psi' \right]. \tag{10b}$$

The value of the wake concentration χ_w is fixed by ensuring that the net flux out of the wake is zero. Therefore, if the wake extends from ϕ_2 to ϕ_3, and if $\psi = 0_+, 0_-$ denotes the upper and lower boundaries of the wake as $\psi \to 0$ for $\psi > 0$, $\psi < 0$, then

$$\int_{\phi_2}^{\phi_3} \left[\left(\frac{\partial\chi}{\partial\psi} \right)_{\psi=0_+} - \left(\frac{\partial\chi}{\partial\psi} \right)_{\psi=0_-} \right] d\phi = 0. \tag{11}$$

Substituting (10) into (11) gives the following estimate for χ_w:

$$\chi_w = \frac{\displaystyle\int_{\phi_2}^{\phi_3} \int_0^\infty \frac{\psi'}{2\sqrt{\pi[(\phi-\phi_2)D]^{3/2}}} \exp\left[-\frac{\psi'^2}{4D(\phi-\phi_2)} \right] \{\chi_{2+}(\psi') + \chi_{2-}(\psi')\} d\psi'\, d\phi}{\displaystyle\int_{\phi_2}^{\phi_3} \int_0^\infty \frac{\psi'}{\sqrt{\pi[(\phi-\phi_2)D]^{3/2}}} \exp\left[-\frac{\psi'^2}{4D(\phi-\phi_2)} \right] d\psi'\, d\phi}. \tag{12}$$

$\pm$ corresponds to $\psi > 0, \psi < 0$ respectively and $A = 4D(\phi - \phi_1)$, $B = 4D(\phi_1 - \phi_s)$.

(iii) *Downstream of the separation point* ($\phi = \phi_2$) the flow has separated from the cylinder. The concentration in and outside the wake can be estimated by *assuming* that the horizontal mixing produced by the vigorous eddying in the wake effectively makes the concentration in the wake (C_w or χ_w) constant. This idea was developed by Puttock and Hunt (1979), and was shown by Puttock (1979) to satisfactorily predict

To evaluate χ_w from (12) the values of ϕ_2, ϕ_3 are required which depend on the flow pattern. Using the free streamline model of Parkinson and Jandali (1970) to describe the inviscid mean flow outside the wake of a circular cylinder (for which the separation point (at say $\theta = 90°$) or base pressure coefficient must be known), the calculation shows $\phi_s = -4.77$, $\phi_1 = -2$, $\phi_2 = 0$. To estimate ϕ_3 we have to know the extent of the recirculating streamlines in the wake, which from the observations we take to be about 3 (normalised on the

Table 1. Estimates of the normalized wake concentration and the ratio of the wake concentration to the maximum surface concentration ($x_s = -4.5R_s$)

y_s/R_s	χ_w	χ_w/χ_{ms}
0.0	3.45	0.64
0.075	2.85	0.84
0.15	1.60	0.64

local radius).

The wake concentration, χ_w, was computed from (2.12) taking the values of ϕ_s, ϕ_1, ϕ_2 mentioned above (see Table 1), and using $D = 0.001$.

If the turbulence in or outside the plume is very small, when the plume travels past the separation line the diffusion primarily results from the eddying in the wake. In such a case a much simpler estimate for wake concentration can be derived based on Gifford's estimates for concentration in wakes behind buildings. One assumes that downwind of the reverse flow region, if the wake thickness is $\lambda_w R_s$, where $\lambda_w \simeq 3$, and if a typical mean velocity across the wake is $U_s/2$, by continuity

$$C \simeq \frac{Q}{(3/2)U_s R_s \sigma_z \sqrt{2\pi}}.$$

In practice, σ_z in the wake is greater than $\sigma_z^{(s)}$, the value in the absence of the hill.

The computed values of the normalised concentration χ are plotted in Fig. 3 as functions of θ the angular position around an axisymmetric hill, and the displacement of the source y_s/R_s, for the parameters relevant to our experiments ($D = 0.001$, $x_s/R_s = -4.5$). In order to appreciate the sensitivity of the result to the lateral displacement of the source it is useful to note that for this value of D the lateral plume width σ_y/R_s is about 0.09, in the absence of the hill. Figure 3 illustrates the following major effects:

(i) The concentration on the centre line of the plume ($z = H_s$) varies very little as the plume is advected around the surface of the hill, in this flow by less than 1/3 between $\theta = 180°$ and $\theta = 90°$, (calculations for a larger D are shown in Fig. 6 of Hunt and Mulhearn, 1973). [In a sense σ_y is enormously increased, but this is perhaps a misleading way of describing the effect (e.g. Start, Ricks and Dickson, 1976).] 1976)].

(ii) The *position* of the maximum surface concentration rapidly moves to the 90° point or the separation line when the source is displaced off the centre line, by a distance that is small compared with R_s but comparable with σ_y at the hill. (In this case $y_s = 0.15R_s$.)

(iii) But the *maximum* surface concentration does

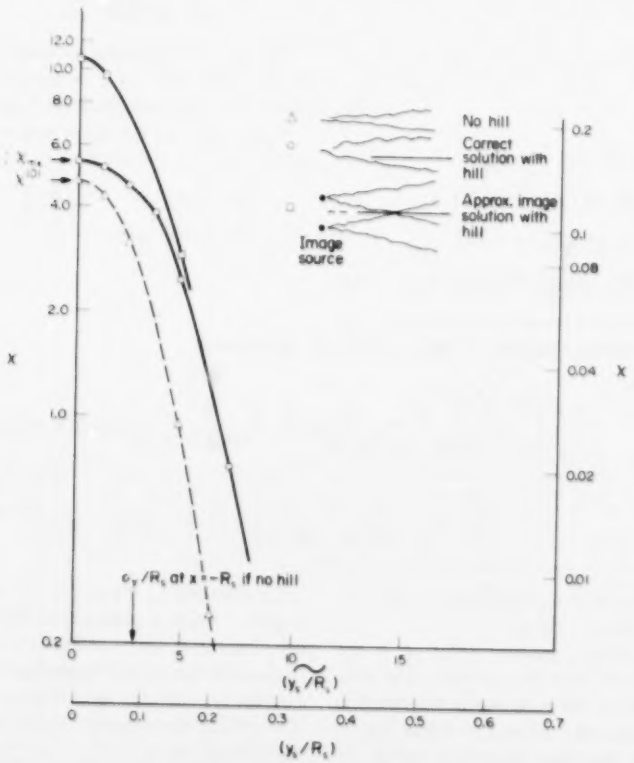


Fig. 4. Maximum concentration χ_{ms} upstream of separation ($\theta \leq 90°$) on the hill as a function of the source offset, y_s/R_s, denoted by ⊙. Points for χ on $\theta = 180°$ in the absence of the hill are denoted by ▽. Points for χ_{ms} based on an image source calculation (only correct for plumes which do not split) are denoted by ⊡.

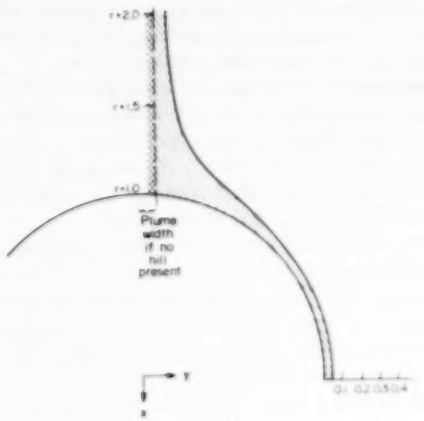
not change so rapidly as y_s/R_s varies.

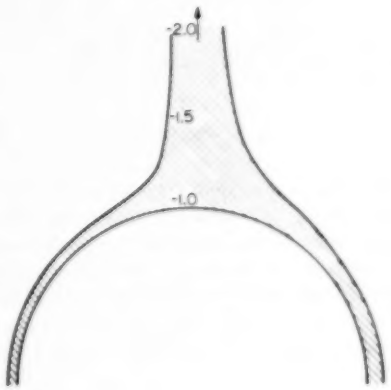
Figure 4 shows the variation of the maximum surface concentration, χ_{ms}, with y_s/R_s for $D = 0.001$ and $x_s = -4.5R_s$. The results are also plotted in a form independent of D as $\tilde{\chi} = \chi_{ms}\sqrt{D}$ as a function of $y_s/(R_s\sqrt{D}) = y_s/\tilde{R}_s$. Note that near the stagnation line the streamlines diverge which leads to *decreased* concentration gradients across streamlines and therefore less diffusion. This process causes the diffusion equation solutions to give a maximum surface concentration somewhat *greater* than the concentration in the centre of the plume in the absence of the hill $\chi^{(\infty)}$ (in fact in proportion to $[(|x_s| - R_s)/(|\phi_1 - \phi_s|)]^{1/2}$. In the present case, where $x_s = -4.5R_s$, on $\theta = 180°$, $\chi_{ms}/\chi^{(\infty)} = 1.12$. The E.P.A. valley model (Burt and Slater, 1977) is based on the image source hypothesis which implies that $\chi_{ms}/\chi^{(\infty)} = 2.0$ whatever the source position along the stagnation line. This implication is certainly an overestimate.

Our theory can lead to the same order of magnitude for χ_{ms} in special circumstances and for a different reason. The ratio $\chi_{ms}/\chi^{(\infty)}$ predicted by our theory is only greater than 2.0 if $x < 2R_s$, for a source on the stagnation line. Because the normal derivative of the concentration at the stagnation line is negligible, no image source is necessary to ensure $\partial C/\partial n = 0$. The reason why the maximum surface concentration is so much higher in the presence of the hill is because the pollution is *advected* on the hillside; in other words it reaches the surface without having to diffuse in a direction normal to the plume. An image source is appropriate, only if such normal diffusion occurs as we see below and in section 3.

On Fig. 4 we have also plotted the "Gaussian Image" which is the maximum surface concentration produced by a source at y_s and an image source at $-(y_s)$. Such a procedure only appears to be correct if $y_s/R_s > 2\sigma_y^{(\infty)}/R_s$, because if $0 < y_s/R_s < 2\sigma_y^{(\infty)}/R_s$, the plume *splits* and a simple image solution is not valid.

In Fig. 5 a plan view of the centre lines and thicknesses of three plumes are shown as they impinge on the hill, as they split and as they are advected round the hill. The outer boundary of the plume is taken for convenience to be $1/e(=1/2.7)$ times the centre line concentration on $x = -1$. These diagrams will be compared with photographs of the experimental low Froude number plumes in Pt. II. The experimental values of $D = 0.001$ was used for the computations.





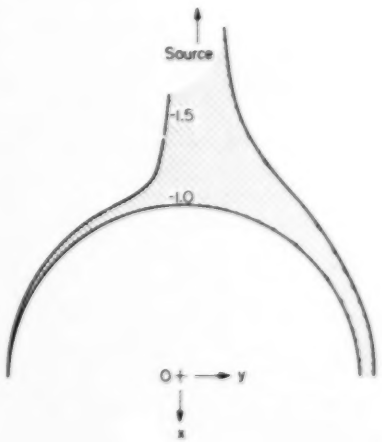


Fig. 5. Outlines of plumes defined by (1/e)x (maximum concentration) at a given value of downstream distance x, (roughly the same as the observed outline of dye plumes), obtained from the diffusion equation when $D = 0.001$. (a) Source at $x_s = -4.5R_s$, $y_s = 0$. (b) Source at $x_s = -4.5_s$, $y_s = 0.037R_s$ (c) Source at $x_s = -4.5R_s$, $y_s = 0.075R_s$.

3. FINITE OR LARGE FROUDE NUMBER

When the Froude number F is greater than 1.0 most streamlines (even those on the centre line) pass over or round the hill and virtually none impinge on the hill. Under such circumstances the pollution reaches the hillside by *diffusion* perpendicular to the centre line of the plume. As Egan and Bass (1976) have pointed out,

in a discussion of this problem, the diffusion from a source in a three-dimensional flow can lead to much higher ground level concentrations than in a two-dimensional flow because the streamlines are much closer to the surface of the hill in the former than the latter case. However, Egan and Bass's (1976) expression for concentration did not result from a solution of the diffusion equation; it was an extrapolation of other Gaussian plume models.

We have developed an exact solution for 3-dimensional diffusion in an *axisymmetric* flow, e.g. potential flow over a hemisphere. This broadly substantiates the estimates of Egan and Bass (1976). For a statistical analysis of turbulent diffusion in a 3-dimensional flow see Hunt, Britter and Puttock (1979).

Consider a system of orthogonal co-ordinates (s, n, γ) fixed relative to the streamline through the source shown in Fig. 6. The advective diffusion equation for a thin plume with its centre line on the meridian $\gamma = \gamma_0$ is then

$$\tilde{U}(s)\frac{\partial C}{\partial s} + n\frac{\partial \tilde{V}}{\partial n}\frac{\partial C}{\partial n} + (\gamma - \gamma_0)\frac{\partial V_\gamma}{\partial \gamma}\frac{\partial C}{\partial \gamma} = D_1(s)\frac{\partial^2 C}{\partial n^2} + D_2(s)\frac{\partial^2 C}{\partial \gamma^2}. \quad (13a)$$

The advective terms containing $\partial\tilde{V}/\partial n$ and $\partial V_\gamma/\partial\gamma$ represent the effects of adjacent streamlines converging or diverging. Since these terms vary quite rapidly near an obstacle, in the strictest sense the equation is only valid for plumes whose thickness σ_n in the normal (n) direction is very much less than the distance over which $\partial\tilde{V}/\partial n$ varies. The conditions for the validity, then, are that upwind of the obstacle

$$\sigma_n \ll R, \quad (13b)$$

where R is distance from the plume centre line to the axis ($y = z = 0$), and over the obstacle

$$\sigma_n \ll n_s, \quad (13c)$$

where n_s is the distance from the plume centre line to the obstacle. We also assume that the diffusivities $D_1(s)$, $D_2(s)$ in the n and γ directions are constant across the plume, but may vary along the plume. This condition is satisfied if (13b) and (13c) are satisfied.

Since the flow is assumed to be axisymmetric $\partial V_\gamma/\partial\gamma = 0$, then by continuity

$$\partial\tilde{V}/\partial n = -\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}s}(\tilde{U}R). \quad (14)$$

By looking for a solution of the form

$$C = C_1 = \frac{1}{g(s)}\,\mathrm{e}^{-[f(s)n^2 + f_2(s)(\gamma - \gamma_0)^2]}, \quad (15)$$

which is subject to the flux across the plume being constant, i.e.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{U}(s)C\,R(s)\,\mathrm{d}n\,\mathrm{d}\gamma = Q, \quad (16)$$

we find

$$C_1 = \frac{Q\exp\{-[\tilde{U}^2R^2n^2/(4\tilde{\phi}(s)) + (\gamma - \gamma_0)^2/(4\tilde{T}(s))]\}}{4\pi\{\tilde{\phi}(s)\tilde{T}(s)\}^{1/2}}, \quad (17)$$

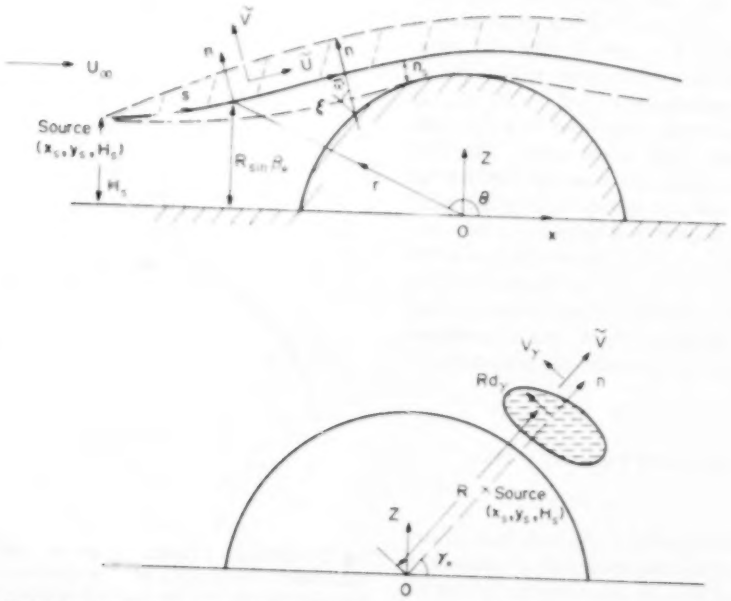


Fig. 6. Definition sketches for analysing diffusion in axisymmetric 3-dimensional potential flow over a hemisphere. (a) Section through the source and the hill. (b) Section through the plume and the hill in the Oyz plane.

where

$$\hat{\phi}(s) = \int_0^s D_1(s')R^2(s')\tilde{U}(s')\mathrm{d}s',$$

$$\hat{T}(s) = \int_0^s \frac{D_2(s')}{R^2(s')\tilde{U}(s')}\mathrm{d}s'.$$

Thus the square of the plume thickness normal to the surface, defined as

$$\int_{-\gamma}^{\gamma}\int_{-\gamma}^{\gamma} n^2 C_1 \,\mathrm{d}n\,\mathrm{d}\gamma \Big/ \int_{-\gamma}^{\gamma}\int_{-\gamma}^{\gamma} C_1 \,\mathrm{d}n\,\mathrm{d}\gamma,$$

is $\sigma_n^2 = 2\hat{\phi}(s)/[\tilde{U}^2(s)R^2(s)]$, and parallel to the surface, in the γ direction, is $\sigma_\gamma^2 = 2R^2(s)\hat{T}(s)$. Very close to the stagnation point, the lowest streamlines diverge strongly, so σ_n increases, but over most of the hill the flow is faster than upstream and the streamlines converge. Then σ_n is reduced below its value over level ground, whereas the effect of the streamlines diverging in the γ direction as the flow passes over the hill increases σ_γ. In Fig. 7 σ_n and σ_γ are computed for the low value ($D = 0.002$) found in our neutral stability experiments to show the typical effects of a 3-dimensional hill on concentration. We have made the same computations for D_1 and D_2 varying along the streamline through the source and, as Hunt and Mulhearn (1973) found for 2-dimensional flows, the same trends appear. Note that the ratio $\sigma_n/\sigma_n^{(-)}$ of the plume thickness over the hill to its value $\sigma_n^{(-)}$ in the absence of the hill is greater than the ratio n_s/H_s of the plume centre line displacement from the hill to the upwind stack height, (in our case for $H_s/h = 0.4$, $\sigma_n/\sigma_n^{(-)} \simeq 1/3$ whereas $n_s/H_s \simeq 1/5$). In other words n_s/σ_n is less than $H_s/\sigma_n^{(-)}$, the implications of which we discuss later.

For practical purposes the analysis needs to be extended to plumes which impinge on the hill, i.e.

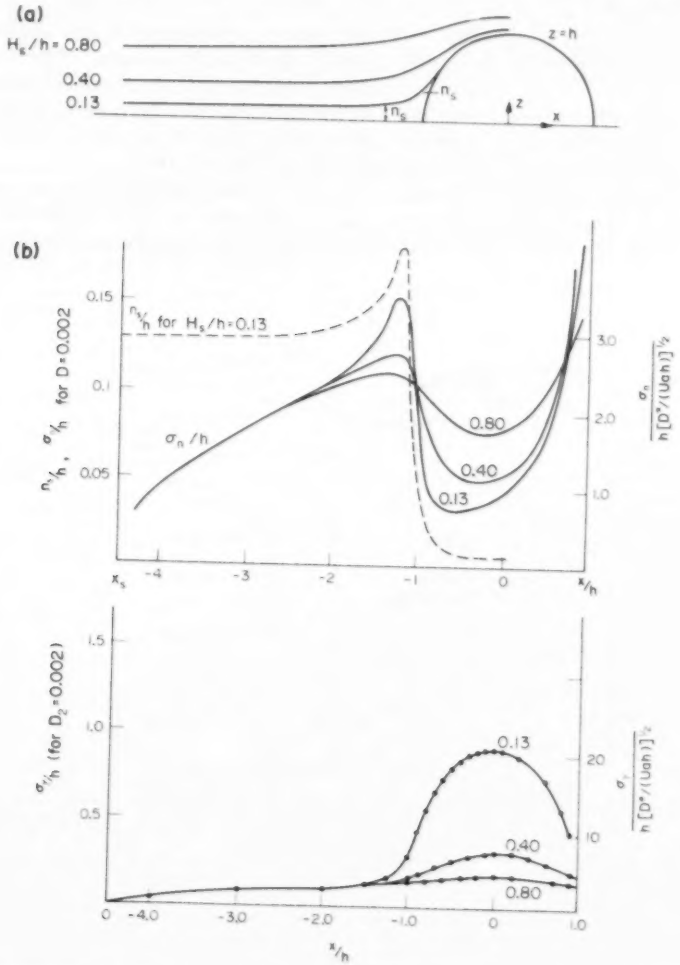


Fig. 7. Calculation of plume trajectories, plume widths and surface concentrations of sources placed at various heights H_s on the centre line of a hemisphere, radius h, assuming potential flow. (a) Plume centre line trajectories for various H_s/h. (b) Plume widths σ_n normal to the streamlines in the xz plane (——) and distance (n_s) from the plume centre line surface to the surface (– – –). (c) Plume width σ_γ in the azimuthal direction i.e. normal to the flow direction.

plumes for which σ_n is of the order of n_s. But for such plumes the condition (13c) is violated, just as it is in the analysis of plumes over level ground where the plume behaviour is predicted on the basis of the velocity and turbulence on the plume centre line. We assume, without proof but on the basis of previous experiments (such as those of Britter, Hunt and Puttock, 1976), and the experiments to be reported in Part II, that C_1 can be estimated near the surface despite violating the condition (13c). But close to the surface, we have to look for an additional concentration distribution ΔC such that the flux of source material normal to the surface at $n = -n_s$ is zero. Thus, if the normal to the surface is at an angle $\bar{\omega}$ to the normal from the streamline, and ξ is a coordinate normal to the surface (Fig. 6a)

$$\frac{\partial C_1}{\partial n}\cos\bar{\omega} + \frac{\partial \Delta C}{\partial \xi} = 0 \quad \text{on } \xi = 0. \tag{18}$$

The flux condition (3.4) becomes

$$\int_{-n_s}^{x}\left\{\int_{-x}^{x}\tilde{U}(s)[C_1 + \Delta C]R(s)\mathrm{d}\gamma\right\}\mathrm{d}n = Q. \tag{19}$$

This boundary condition is only significant when the surface concentration is appreciable, which implies $n_s \sim \sigma_n$. Since σ_n is assumed to be small compared to R, then it implies that the boundary condition (18) is only significant when $n_s \ll R$. Consequently $\cos\bar{\omega} = 1 - 0[n_s^2/R^2]$, and since terms of $0(n_s^2/R^2)$ have been neglected in the analysis, the assumption

$$\partial \Delta C/\partial \xi = \partial \Delta C/\partial n = -\partial C_1/\partial n \tag{20}$$

is consistent. Then the solution for ΔC satisfying (13a), (19) and (20) can be found by using the facts (a) where the plume impinges, the product $(\tilde{U}Rn_s)$ is constant along the plume and (b) that terms of $0(n^2/R^2)$ are negligible. The simple "image" solution results, so

$$C = C_1 + \Delta C$$

where

$$\Delta C(s,n,\gamma) = \frac{Q\exp[-\{\tilde{U}^2R^2(n+2n_s)^2/(4\hat{\phi}(s)) + (\gamma-\gamma_0)^2/(4\hat{T}(s))\}]}{4\pi[\hat{\phi}(s)\hat{T}(s)]^{1/2}}. \tag{21}$$

and where C_1 is given by (17). This solution cannot be strictly applied near a stagnation point or near the plane $y = 0$ if the source is upwind of the hill. However the solution does not become singular there and it gives useful estimates of concentration (see Hunt, Britter and Puttock, 1979).

In Fig. 8 computations of surface concentrations over a hemispherical hill are plotted for various heights (H_s/h) of an upstream source $(x_s/h = -4.5)$.

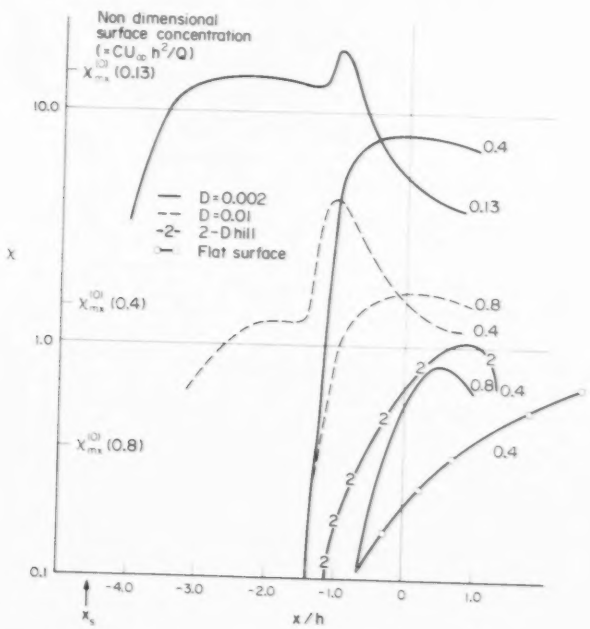


Fig. 8. Surface concentrations over a hemisphere (height h) on the plume centre line for two diffusivities (— — $D = 0.002$) (— · — $D = 0.01$) and various source heights (H_s). Upstream source is at $x_s = -4.5h$. For reference the surface concentration over a half cylinder and over a flat surface when $H_s/h = 0.4$ are plotted, and the values of maximum surface concentration $\chi^{(0)}_{mx}$ over flat surfaces are indicated on the χ axis.

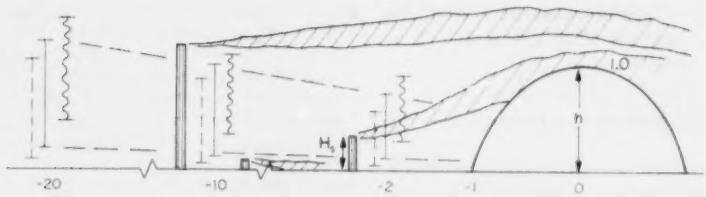


Fig. 9. Potential flow over a hemisphere with constant diffusivity, D along each plume. Schematic demonstration of the "window" or range of values of H_s/h for which the maximum ground level concentration is increased by at least a factor of 2. If H_s is too large the plume misses the hill; if too small the plume touches the ground upstream. Note that D is chosen so that σ_z at $x = 0$ is equal to its value in a neutral atmosphere, and $h \simeq 500$ m. This is the window *1*. If D is increased the "window" is raised ⟿, if D is decreased the "window" is lowered |—|.

We assume $D_1 = D_2 = 0.002$.* Note (i) the factor of ten reduction of the maximum surface concentration as H_s/h is increased from 0.4 to 0.8 and (ii) the high ratio of the maximum concentration on the surface of the hill relative to its maximum value on level ground in the absence of the hill. For the case with the source height at $0.4h$ this ratio is about 5.5. (For potential flow over a *two-dimensional* semi-circular hill the comparable ratio is about 0.75 – a result obtained from Hunt and Mulhearn (1973, equation 3.13), and approximately confirmed by recent wind tunnel experiments by Britter and Hunt (to be published).

These two qualitative observations can be generalised by inspection of (21) and (17). The most interesting case is where s_1 the distance from the source to the start of the hill (say where its elevation is about 1/10 of its maximum), is small enough that the part of the plume path above the hill contributes much more to the integral

$$\hat{\phi}(s) = \int_0^s D_1 R_2 \tilde{U}\, ds$$

than does the part upstream, i.e.

$$H_s^2 s_1 \ll h^2 L \quad \text{and so} \quad \hat{\phi}(s_h) \sim D_1 h^2 L U_r, \tag{22a}$$

where L is the half length of the hill, and s_h is the value of s at the top of the hill. We also assume that the source is high enough that the location of the maximum ground level concentration is downwind of the hill, i.e. if D_1 is constant,

$$H_s^2 \gg s_h D_1 / U_r. \tag{22b}$$

From continuity and the axisymmetry of the flow

$$2\pi \tilde{U}(s) R(s) n_s = \pi U_r H_s^2. \tag{23}$$

Therefore, given (22), a rough estimate for the variation of the maximum surface concentration on the hill with source height is [over the limited range of source heights satisfying 22(a) and 22(b), shown in Fig. 9]

$$C_{mx}(n = -n_s) \sim C^{(\infty)}(s_h) \frac{H_s}{h} \times \exp\{-(H_s/h)^4/(16DL/(\tilde{U}h^2))\},$$

$$\text{or } C_{mx}(n = -n_s) \sim C^{(\infty)}(s_h) \frac{H_s}{h} \times \exp\{-(H_s/h)^4/[2(\sigma_n/h)^2 \times (\tilde{U}(s_h)/U_r)^2]\}, \tag{24}$$

where $C^{(\infty)}$ is the centre line concentration of the plume in the absence of the hill. This result is quite different from the equivalent result for a two-dimensional hill or a plane where $C_{mx}(n = -n_s)$ is inversely proportional to the square of the source height (H_s^2). The maximum surface concentration on a two-dimensional hill is not closely related to $C^{(\infty)}$ because such a hill does not really "penetrate" into a plume in the same way.

An important implication of (24) is that the variation of concentration on a 3-dimensional hill is primarily determined by the *displacement* of the streamlines towards the surface, and only slightly by the convergence and divergence of the streamlines. This diffusion process is somewhat different from that in two-dimensional flows where the convergence approximately cancels out the effect of streamlines being displaced towards the surface. The reason is that the *stronger* convergence in 3-dimensional flows induces a *stronger* normal diffusion across the streamlines (see Keffer, Kawall, Hunt and Maxey, 1978); so the net effect is that the crucial ratio n_s^2/σ_n^2 is much smaller in 3-dimensions than in 2-dimensions.

Another way of expressing (22a) and (22b) is to say that for each value of s there is a *limited* range of H_s for which a significant amplification (which we *define* as a factor of 2) of the maximum ground level concentration is observed. As we have explained this amplification can only occur if the position of C_{mx} lies on or near the hill location. If H_s is smaller than this range the position of C_{mx} lies upwind of the hill and for H_s too large, C_{mx} lies well beyond the hill. In either case there is little amplification. This "window" of critical H_s values has been computed from (21) and (17) and is depicted in Fig. 9 for a hill 500 m high, where D is chosen so that in the absence of the hill the calculated

* All the following conclusions based on these computations are much the same if D_1 and D_2 are allowed to vary with s_1 as they do in practice.

σ_z, σ_y agree with typical values that are observed in the atmosphere. So for example when $x/h = 5$, $\sigma_z/h \simeq 0.25$, and $D^*/(Uh) \simeq 0.006$.

4. IMPLICATIONS OF THE THEORY

Part II will show how the theory agrees with the results of laboratory experiments which were designed to correspond approximately to the theoretical conditions. However the theory developed here can also be and is being applied to atmospheric and other turbulent diffusion problems. A few words of comment are appropriate:

(i) The results of section 2.2 showed that the plume behaviour is very sensitive to quite small changes in wind direction away from the direction that transports the plume on to a stagnation point. The analysis provides a simple way to estimate the effect of these changes.

(ii) The analysis of section 2.2 can also be applied to diffusion in a *turbulent* flow from a line source around a circular cylinder when the source is displaced from the stagnation line. Wind tunnel experiments of Britter have been compared with this calculation, and will be published soon.

(iii) Regarding the applicability of the analysis of section 2 to chimney plumes, Brighton (1978), and Burt and Slater (1977) have described plumes impinging on 3-dimensional hillsides, and have observed high concentrations of the order of those in the centre of the plume.

(iv) The analysis for neutral flow over a hemisphere is capable of approximate but simple generalization to ellipsoidal hill shapes because we have found that the primary effect of a three dimensional hill is to displace the streamlines closer to the hill surface, the effects of convergence and divergence of streamlines being quite small.

Acknowledgements – In the writing of this paper J. C. R. Hunt was supported by a grant from the U.S. Environmental Protection Agency to the North Carolina State University Department of Geosciences, R805595. We are grateful to Mr. Julian Smith and Dr. T. Gopal for help with some of the computing. Dr. Isaacs of Environmental Research and Technology, Concord, Mass., kindly provided the computations of Fig. 9.

REFERENCES

Abramowitz M. and Stegun I. A. (1964) *Handbook of Mathematical Functions*, pp. 297, 319. Dover, N.Y.

Barratt R. V. (1973) Use of the wind tunnel to investigate the influence of topographic features on pollution from a tall stack. Paper presented at the Chimney Design Symp., Centre for Industrial Consultancy & Liaison, Univ. of Edinburgh.

Brighton P. W. M. (1978) Strongly stratified flow past three-dimensional obstacles and mesoscale vortex streets in the atmosphere. *Q. Jl R. Met. Soc.* **104**, 289.

Britter R. E., Hunt J. C. R. and Puttock J. S. (1976) Predicting pollution concentrations near buildings and hills. *Proc. Conf. Systems and Models in Air and Water Pollution*. Inst. Meas. and Control, London.

Burt E. W. and Slater H. H. (1977) Evaluation of the valley model. Paper presented at AMS-APCA Joint Conf. on Appl. of Air. Pollut. Met., Utah, Nov–Dec.

Drazin P. G. (1961) On the steady flow of a fluid with variable density past an obstacle. *Tellus* **XIII**, 239–251.

Egan B. (1975) Turbulent diffusion in complex terrain. Presented at Am. Met. Soc. Workshop on Air Pollution Met. & Env. Assess.

Egan B. and Bass A. (1976) Air quality modelling of effluent plumes in rough terrain. *Proc. 3rd Am. Met. Soc. Meeting on Atm. Diff. & Air. Qual.* Raleigh.

Eliseev V. S. (1975) Trajectories of smoke jets from industrial sources TR. G.G.O. 373. EPA transl. TR-77-235.

Huber A. H., Snyder W. H., Thompson R. S. and Lawson R. E. (1976) Stack placement in the lee of a mountain ridge, a wind tunnel study. EPA 1600/4-76-047.

Hunt J. C. R. (1973) A theory of turbulent flow around 2-dimensional bluff bodies. *J. Fluid Mech.* **61**, 625–.

Hunt J. C. R., Abell C. J., Peterka J. A. and Woo H. C. (1978) Kinematical studies of flow around free or surface mounted obstacles; applying topology to flow visualisation. *J. Fluid Mech.* **86**, 179–200.

Hunt J. C. R., Britter R. E. and Puttock J. S. (1979) Mathematical models of dispersion of air pollution around buildings and hills. *Proc. I.M.A. Sympm. Maths. Models of Turbulent Diffusion in the Environment*, 1978. Acad. Press.

Hunt J. C. R. and Mulhearn P. J. (1973) Turbulent dispersion from sources near two-dimensional obstacles. *J. Fluid Mech.* **61**, 245–274.

Hunt J. C. R. and Snyder W. H. (1979) Experiments on stably and neutrally stratified flow over a model 3-dimensional hill. *J. Fluid Mech.* (submitted).

Hunt J. C. R., Snyder W. H. and Lawson R. E. (1979) Flow structure and turbulent diffusion around a 3-dimensional hill; fluid modelling study on effects of stratification Pt. I. Flow structure. U.S. E.P.A. Monit. Ser. Rep. EPA-600/4-78-041.

Keffer J. F., Kawall J. A., Hunt J. C. R. and Maxey M. R. (1978) The uniform distortion of thermal and velocity mixing layers. *J. Fluid Mech.* **86**, 456–490.

Parkinson G. V. and Jandali T. (1970) A wake source model for bluff body potential flow. *J. Fluid Mech.* **40**, 577–.

Puttock J. S. (1979) Turbulent diffusion from sources near obstacles with separated wakes. Pt. II Concentration measurements near a circular cylinder in uniform flow. *Atmospheric Environment* **13**, 15–22.

Puttock J. S. (1976) Turbulent diffusion in separated and in stratified flows. Ph.D. diss., Univ. Cambridge.

Puttock J. S. and Hunt J. C. R. (1979) Turbulent diffusion from sources near obstacles with separated wakes. Pt. I An eddy diffusivity model. *Atmospheric Environment* **13**, 1–13.

Riley J. J., Lin H. T. and Geller E. W. (1976) A numerical and experimental study of stable stratified flow around complex terrain. U.S. E.P.A. Env. Mon. Ser. Rept. EPA-600/4-76-021.

Start G. E., Ricks N. R. and Dickson C. R. (1976) Effluent dilutions over mountainous terrain. *Proc. 3rd Am. Met. Soc. Meeting on Atm. Diff. and Air Qual.*, Raleigh.

APPENDIX

If we denote the solution derived in the text by $\chi^{(1)}$, this is the solution of

$$\partial\chi^{(1)}/\partial\phi = D\frac{\partial^2\chi^{(1)}}{\partial\psi^2}. \qquad \text{(A1)}$$

Near $(\phi, \psi) = (\phi_1, 0)$, however, the assumption (5) does not hold, and the full Equation (4) must be used. Subtracting (A1) from (4) gives

$$\frac{\partial \chi'}{\partial \phi} - D\frac{\partial^2 \chi'}{\partial \psi^2} - D\frac{\partial^2 \chi'}{\partial \phi^2} = D\frac{\partial^2 \chi^{(1)}}{\partial \phi^2}, \qquad \text{(A2)}$$

where $\chi' = \chi - \chi^{(1)}$ is a correction to be calculated near $(\phi_1, 0)$. The right hand side of this equation is known. Taking a Fourier transform in ϕ leads to an ordinary differential equation in ψ, from which the unknown boundary values

$$\frac{\partial \chi'}{\partial \psi} \quad \text{on} \quad \psi = 0, \quad \phi < \phi_1$$

and

$$(\chi'_+ - \chi'_-) \quad \text{on} \quad \psi = 0, \quad \phi > \phi_1$$

can be calculated by the Wiener–Hopf technique (Puttock, 1976) for the region where $(\phi - \phi_1) \ll (\phi_1 - \phi_s)$. Since it can be shown that $\chi'_+ + \chi'_- = 0$ on $\psi = 0$, the correction on the upper and lower sides of the impervious boundary, i.e. the surface near the stagnation point, can be calculated. The result is

$$\chi'(\phi, 0_+) = -\frac{\beta}{D\pi^{3/2}} e^{-(\phi_1 - \phi_s)\beta^2/D} \times$$

$$\times \int_0^x \left(\frac{y}{(y+1)|y-\beta^2|}\right)^{1/2} d(\eta) g_I(y, \beta^2) dy, \qquad \text{(A3)}$$

where

$$\beta = \frac{\psi_s}{2(\phi_1 - \phi_s)}, \quad \eta = \sqrt{(\phi - \phi_1)y/D}$$

$$g_I(y, \beta^2) = (-i)\mathrm{Im}\{w(\sqrt{(\phi_1 - \phi_s)/D}$$
$$(\sqrt{y(y+1)} + i\sqrt{\beta^2 - y}))\} \quad \text{for } y < \beta^2$$

$$g_I(y, \beta^2) = \exp\{-\frac{(\phi_1 - \phi_s)}{D} \times$$
$$(y^2 + 2y - \beta^2 + 2\sqrt{(y+1)y(y-\beta^2)})\} \quad y > \beta^2$$

and $w(\ )$, $d(\eta)$ are the probability integral and Dawson's integral, respectively, as defined by Abramowitz and Stegun (1964).

Computation of (A3) shows that the correction is very small. No point in Fig. 3 would be changed by more than 0.01%.